A STUDY OF THE HERMITIAN NUMERICAL METHOD APPLIED TO THE SINGLE DEGREE OF FREEDOM, DAMPED OSCILLATOR

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ABSTRACT

Approximation of the differential equation of the single degree of freedom, damped oscillator by the first order central finite difference equations and two Hermitian finite difference equations is investigated. An error analysis is made between the solution to the differential equation and the solutions to the finite difference approximations for various values of damping. The results of the error analysis indicate that the Hermitian approximations have less error than the first order difference approximation for the same size of increment. Furthermore, the employment of the Hermitian method does not materially increase the execution time on the digital computer over that required by the first order difference equations. Thus the Hermitian equations are superior to the first order finite difference equations for the damped oscillator problem.

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TABLE OF SYMBOLS

A	arbitrary constant (in)
c	viscous damping coefficient (lb_f sec/in)
e	base of the natural system of logarithms (2.71828)
i	$\sqrt{-1}$
j	finite difference station number
k	spring constant (lb_f/in)
m	mass $\left(\frac{lb_f\ sec^2}{in}\right)$
r	$\sqrt{x^2+z^2}$ real part of λ where $\lambda = re^{i\theta}$ (dimensionless)
r_c	r for conventional method
t	time (seconds)
x	real part of λ where $\lambda = x + iz$ (dimensionless)
x_c	x for conventional method
y	displacement (in)
z	imaginary part of λ where $\lambda = x + iz$ (dimensionless)
z_c	z for conventional method
α	characteristic complex root in the solution of the exact equation (dimensionless)
α_i	imaginary part of α
α_r	real part of α
Δ	finite difference interval (space or time increment)
Δt	finite difference step in time (sec/time interval)
ζ	damping ratio
θ	arc sin $\frac{z}{r}$ dimensionless frequency factor
θ_c	θ for conventional method
θ_1	θ for MOD 1 method

θ_2 — θ for MOD 2 method

κ — $-\ln r$ dimensionless attenuation factor

κ_c — κ for conventional method

κ_1 — κ for MOD 1 method

κ_2 — κ for MOD 2 method

λ — characteristic complex root in the solution to the finite difference equations (dimensionless)

λ_c — λ for conventional method

λ_1 — λ for MOD 1 method

λ_2 — λ for MOD 2 method

μ — $\dfrac{\theta}{a_i \Delta t}$ relative error in θ

μ_c — μ for conventional method

μ_1 — μ for MOD 1 method

μ_2 — μ for MOD 2 method

ν — $\dfrac{\kappa}{a_r \Delta t}$ relative error in κ

ν_c — ν for conventional method

ν_1 — ν for MOD 1 method

ν_2 — ν for MOD 2 method

ω — natural frequency of vibration (rad/sec)

$\bar{\omega}$ — $\omega \Delta t$, sampling resolution (rad/time interval)

CHAPTER I

INTRODUCTION

In the quest for accurate numerical solutions to differential equations, considerable effort has been devoted to the determination of the errors associated with various finite difference schemes. This work has gained importance with the recent widespread use of the digital computer. Due to the high value placed on computer time and to the limitations on core storage space within the computer, the accuracy and efficiency of the numerical method employed are of the utmost importance.

In general, many engineering problems can be classified as either boundary value problems or initial value problems.[2] For the numerical solution to boundary value problems, the most common method used is a form of the central finite difference operators. Several forms of these operators can be found in Table III of the Appendix to reference 1. Some of these are frequently used to solve engineering problems while others seem either to be impractical or to appear seldom in current literature. The Hermitian equations, which are included in this table, fall into the last category. For initial value problems, numerical methods of solution such as Adams, Runge-Kutta, and other forms of the predictor-corrector methods are usually employed. The Hermitian method is also applicable to initial value problems but is seldom used.

The purpose of this paper is to evaluate the merits of the Hermitian method when applied to a second order, ordinary, linear differential equation. This evaluation will be made by comparing the solution to the exact differential equation with the solution to two Hermitian finite difference approximations and with the solution to the first order central finite difference equation. In this manner, the difference in accuracy between these finite difference methods can be determined, and the merits of the Hermitian method established.

The equation chosen for this analysis is that of the single degree of freedom, damped oscillator. This equation was chosen because it contains both the first and second derivatives and because its exact solution is well known.[4,5] This appears to be the first application of the Hermitian equations to differential equations containing more than one degree of derivatives. This problem is generally considered an initial value problem. However, the results of this analysis are also applicable to boundary value problems since a boundary value can be solved using methods of solution normally employed in initial value problems. Also, the oscillator may become a boundary value problem if the displacement or velocity is specified at a time other than zero. Thus, the choice of this example allows the conclusions to be applied to both initial value and boundary value problems.

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CHAPTER II

DISCUSSION OF THE CENTRAL FINITE DIFFERENCE METHOD AND ITS ASSOCIATED ERRORS

There are three types of error associated with the finite difference solution to engineering problems.[3] One type is called truncation or discretization error. The conversion of a differential equation to an algebraic equation is, in essence, the approximation of a continuous system by a discrete mode. The error involved in this conversion is called the truncation error. A graphic example of how this error is introduced can be seen by examining the derivation of the first order central finite difference equations for the first and second derivatives presented in Appendix A.

In the derivation of the finite difference expression for the first derivative, only the first three terms of the Taylor series are used. The first four terms are used in the expression for the second derivative. The truncation error can be approximated by the first unused term in the Taylor series. Therefore, a decrease in truncation error can be accomplished by using more terms of the Taylor series in the finite difference equation.

Another type of error is called round-off error. This error is generated within the computer and is a function of the amount of calculation required and the condition of the numbers involved. Each arithmetic computation, such as addition, multiplication, and exponentiation, has a round-off error associated with it. The computer is limited to carrying a specified number of digits. Any amount over this number must be dropped for each calculation. To decrease round-off error, it is essential to minimize the number of arithmetic calculations within the program and to avoid subtraction of large numbers, etc.

The third type of error is called inherent error. This error, unlike truncation error and round-off error, is not caused by the use of numerical methods.[3] Inherent error is introduced into the program within the data. If the data are not exact, which is usually the case, the solution cannot be expected to be exact.

The object of any numerical analysis is to obtain as accurate an answer as possible without significantly increasing the computation time. There are a number of higher order finite difference methods available that have less truncation error. Hence, they provide more accurate solutions than the first order central finite difference method. Among these is the Hermitian method. These methods vary by the number of stations used, the location of the stations, etc.

When central finite difference methods are applied to an engineering problem, the problem is reduced to that of solving a matrix. This matrix will usually be a band matrix centered about the principal diagonal. The width of the band is dependent upon the number of stations included in the finite difference equation. Using the first order difference equations listed in Appendix A, the band width would be three, and the matrix is called a tridiagonal matrix. If higher order difference equations that require more stations are used, the band width widens. By widening the band of the matrix, more computations may be required than for the tridiagonal matrix. This increases the round-off error and the computation time. Furthermore, the tridiagonal matrix will require less core storage than the wider band matrix for the same element size.

Another approach used to decrease the magnitude of the truncation error involves decreasing the interval between the stations. This is equivalent to increasing the degree of the matrix and, therefore, introduces greater round-off error. Other effects of this procedure are the increase in computer storage space required and the increase in computer running time.

The Hermitian method considered in this thesis results in a tridiagonal matrix but also decreases the truncation error. Consequently, the gain in accuracy due to the decrease in truncation error is not offset by an increase in round-off error. Furthermore, this method can be used to advantage on those problems where core storage requirements are important.

CHAPTER III

APPLICATION OF THE HERMITIAN METHOD TO THE DIFFERENTIAL EQUATION

The first order central finite difference equations for the first and second derivative, as derived in Appendix A, are

$$y_j' = \frac{1}{2\Delta}\left(y_{j+1} - y_{j-1}\right) \tag{3.1a}$$

$$y_j'' = \frac{1}{\Delta^2}\left(y_{j+1} - 2y_j + y_{j-1}\right) \tag{3.1b}$$

The Hermitian equations for the first and second derivatives, as derived in Appendix B, are

$$\left(y_{j+1} - y_{j-1}\right) = \frac{\Delta}{3}\left(y_{j+1}' + 4y_j' + y_{j-1}'\right) \tag{3.2a}$$

$$\left(y_{j+1} - 2y_j + y_{j-1}\right) = \frac{\Delta^2}{12}\left(y_{j+1}'' + 10y_j'' + y_{j-1}''\right) \tag{3.2b}$$

where (') denotes the derivative with respect to time.

As shown in Appendix B, the Hermitian equations have less truncation error than the first order equations. Note, that in order to get this reduction in trucation error it was not necessary to use more stations.

Consider the single degree of freedom, damped oscillator shown in Figure 1. The differential equation of the oscillator is [4,5]

$$my'' + cy' + ky = 0 \tag{3.3}$$

where m is the mass of the system, c is the viscous damping coefficient, k is the spring constant, and y is the displacement. Substituting the conventional first order central difference equations, Eqs. 3.1a and 3.1b, for the first and second derivatives into Eq. 3.3 leads to

$$\frac{m}{\Delta t^2}\left(y_{j+1} - 2y_j + y_{j-1}\right) + \frac{c}{2\Delta t}\left(y_{j+1} - y_{j-1}\right) + ky_j = 0 \tag{3.4}$$

where Δ has been replaced with Δt to denote the time interval.

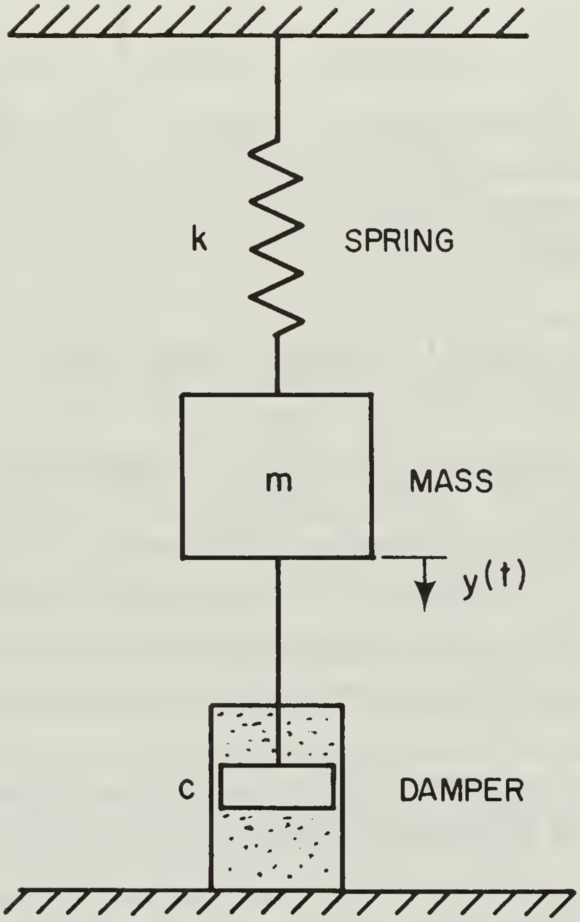

FIGURE 1 THE SINGLE DEGREE OF FREEDOM, DAMPED OSCILLATOR

Since the Hermitian equations are not of the same form as the conventional equations, a different method of approximation to Eq. 3.3 is necessary. Examination of the right side of Eqs. 3.2a and 3.2b reveals that Eq. 3.2a contains the first derivative at the three stations j+1, j, and j-1. Likewise, the right side of Eq. 3.2b contains the second derivative at the same three stations. By adding a third equation expressing y at j+1, j, and j-1 and modifying one of the Hermitian equations, it is possible to solve the differential equations at these three stations simultaneously. Equations 3.5, 3.6, and 3.7 show this procedure.

$$\frac{m}{\Delta t^2}\left(y_{j+1} - 2y_j + y_{j-1}\right) = \frac{m}{12}\left(y''_{j+1} + 10y''_j + y''_{j-1}\right) \tag{3.5}$$

$$\frac{c}{4\Delta t}\left(y_{j+1} - y_{j-1}\right) = \frac{c}{12}\left(y'_{j+1} + 4y'_j + y'_{j-1}\right) \tag{3.6}$$

$$\frac{k}{12}\left(y_{j+1} + 10y_j + y_{j-1}\right) = \frac{k}{12}\left(y_{j+1} + 10y_j + y_{j-1}\right) \tag{3.7}$$

Adding Eqs. 3.5, 3.6, and 3.7 gives

$$\frac{m}{\Delta t^2}\left(y_{j+1} - 2y_j + y_{j-1}\right) + \frac{c}{4\Delta t}\left(y_{j+1} - y_{j-1}\right) + \frac{k}{12}\left(y_{j+1} + 10y_j + y_{j-1}\right) = \frac{10m}{12}y''_j + \frac{4c}{12}y'_j + \frac{10k}{12}y_j \tag{3.8}$$

Note that the right side of Eq. 3.8 will equal zero if $\frac{6c}{12} y'_j$ is added to both sides. The conventional approximation for y'_j gives

$$\frac{c}{4\Delta t}\left(y_{J+1} - y_{J-1}\right) = \frac{c}{12} y'_J$$

(3.9)

If Eq. 3.9 is added to Eq. 3.8, the result is

$$\frac{m}{\Delta t^2}\left(y_{J+1} - 2y_J + y_{J-1}\right) + \frac{c}{2\Delta t}\left(y_{J+1} - y_{J-1}\right)$$

$$+ \frac{k}{12}\left(y_{J+1} + 10y_J + y_{J-1}\right) = 0$$

(3.10)

This is one form of the Hermitian approximation to Eq. 3.3.

To get the second form of the approximation, it is necessary to write Eq. 3.7 as

$$\frac{k}{6}\left(y_{J+1} + 4y_J + y_{J-1}\right) = \frac{k}{6}\left(y_{J+1} + 4y + y_{J-1}\right)$$

(3.11)

When Eqs. 3.5, 3.6, and 3.11 are added, then

$$\frac{2m}{\Delta t^2}\left(y_{J+1} - y_J + y_{J-1}\right) + \frac{c}{2\Delta t}\left(y_{J+1} - y_{J-1}\right)$$

$$+ \frac{k}{6}\left(y_{J+1} + 4y_J + y_{J-1}\right) = \frac{10m}{6} y''_J + \frac{4c}{6} y'_J + \frac{4k}{6} y_J$$

(3.12)

Note that the right side of Eq. 3.12 will equal zero if my''_j is subtracted from both sides. From the conventional approximation for y''_j,

$$\frac{m}{\Delta t^2}\left(y_{J+1} - 2y_J + y_{J-1}\right) = my''_J \qquad (3.13)$$

If Eq. 3.13 is subtracted from Eq. 3.12, the result is

$$\frac{m}{\Delta t^2}\left(y_{j+1} - 2y_j + y_{j-1}\right) + \frac{c}{2\Delta t}\left(y_{j+1} - y_{j-1}\right) + \frac{k}{6}\left(y_{j+1} + 4y_j + y_{j-1}\right) = 0 \tag{3.14}$$

This is the second form of the Hermitian approximation to Eq. 3.3. In this thesis, Eq. 3.10 will be called MOD 2, and Eq. 3.14 will be called MOD 1.

By following the derivation of these two Hermitian approximations, it should be clear that a first order central finite difference equation for either y'_j or y''_j, was used in each derivation. For example, the first order central finite difference equation for y'_j, Eq. 3.9, was introduced in the derivation of Eq. 3.10, the MOD 2 approximation. This is equivalent to modifying Eq. 3.6 to be

$$\frac{c}{2\Delta t}\left(y_{j+1} - y_{j-1}\right) = \frac{c}{12}\left(y'_{j+1} + 10y'_j + y'_{j-1}\right) \tag{3.15}$$

Appendix C shows that the truncation error in Eq. 3.15 is the same as the truncation error in the first order central finite difference equation for the first derivative, Eq. 3.9. Thus, when the Hermitian equations are applied to the differential equation containing more than one degree of derivative, accuracy can be gained for only one specified derivative at a time. The MOD 1 approximation has the accuracy gain in the first derivative, and the MOD 2 approximation has the accuracy gain in the second derivative.

Examination of Eqs. 3.4, 3.10, and 3.14 reveals that the only difference in these three equations is in the third term. Equations 3.10 and 3.14 allow values for y_{j+1} and y_{j-1}, while Eq. 3.4 does not. This ability to let y vary over the interval j+1 to j-1 is the reason truncation error is decreased. If the variation of y happens to be linear over the three successive stations j+1, j, and j-1, Eqs. 3.10 and 3.14 then both reduce to the conventional equation, Eq. 3.4.

CHAPTER IV

COMPARISON OF THE SOLUTIONS

The three finite difference approximations to the differential equation, Eq. 3.3, are repeated here for convenience.

Conventional

$$\frac{m}{\Delta t^2}\left(y_{J+1} - 2y_J + y_{J-1}\right) + \frac{c}{2\Delta t}\left(y_{J+1} - y_{J-1}\right) + k y_J = 0 \qquad (3.4)$$

MOD 2

$$\frac{m}{\Delta t^2}\left(y_{J+1} - 2y_J + y_{J-1}\right) + \frac{c}{2\Delta t}\left(y_{J+1} - y_{J-1}\right) + \frac{k}{12}\left(y_{J+1} + 10y_J + y_{J-1}\right) = 0 \qquad (3.10)$$

MOD 1

$$\frac{m}{\Delta t^2}\left(y_{J+1} - 2y_J + y_{J-1}\right) + \frac{c}{2\Delta t}\left(y_{J+1} - y_{J-1}\right) + \frac{k}{6}\left(y_{J+1} + 4y_J + y_{J-1}\right) = 0 \qquad (3.14)$$

The exact solution to the differential equation is of the form

$$y = A\, e^{\alpha t} \qquad (4.1)$$

where A is an arbitrary constant. Substituting Eq. 4.1 into the differential equation, Eq. 3.3, leads to

$$\left(m\alpha^2 + c\alpha + k\right) A\, e^{\alpha t} = 0$$

If Eq. 4.1 is to be valid for all values of t, then

$$m\alpha^2 + c\alpha + k = 0$$

Thus,

$$\alpha = \frac{\left(-c \pm \sqrt{c^2 - 4mk}\right)}{2m} \qquad (4.2)$$

If $\omega = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2m\omega}$, then Eq. 4.2 becomes

$$\alpha = -\omega\zeta \pm i\omega\sqrt{1-\zeta^2} \tag{4.3}$$

ζ is the nondimensional damping coefficient. A value of ζ less than 1.0 means the system is underdamped, and the motion is oscillatory. A value of ζ greater than 1.0 means the system is overdamped, and the motion is non-oscillatory. When ζ is exactly equal to 1.0, the system is critically damped. For this analysis, the special case of critical damping is not considered.

The solution assumed for the finite difference equations is[2]

$$y_J = A\,\lambda^J \tag{4.4}$$

where j is the station number. Solutions for λ that satisfy Eqs. 3.4, 3.10, and 3.14 are derived in Appendix D. For the first order finite difference equation,

$$\lambda = \frac{\left(1-\frac{\bar{\omega}}{2}\right) \pm i\bar{\omega}\sqrt{1-\zeta^2-\frac{1}{4}\bar{\omega}^2}}{\left(1+\zeta\bar{\omega}\right)} \tag{4.5}$$

where $\bar{\omega} = \omega\Delta t$.

A discussion of $\bar{\omega}$ is in order here. The quantity ω is the natural frequency of the oscillator in radians per second. Therefore, $\omega\Delta t$ or $\bar{\omega}$ has the units of radians per time interval and is a measure of the resolution of the solution. A small value of $\bar{\omega}$ would mean that the system is being sampled many times per cycle. For example, a value of $\bar{\omega}$ equal to 0.5 would mean the system was being sampled every 0.5 radians, or over 12 times per cycle. A value of $\bar{\omega}$ over 1.0 would mean that the solution has a very poor resolution, and the truncation error would be high.

Equation 4.1 can be written in the form

$$y_J = A\,e^{\alpha(J\Delta t)}$$

or

$$y_J = A\left[e^{\alpha\Delta t}\right]^J \tag{4.6}$$

Comparing Eqs. 4.4 and 4.6 reveals that

$$e^{a\Delta t} \quad vs \quad \lambda$$

is a proper comparison. However, Eqs. 4.3 and 4.5 show that both a and λ are complex numbers for the underdamped system. Therefore,

$$e^{a\Delta t} = e^{a_i \Delta t} e^{\pm i a_r \Delta t} \tag{4.7}$$

where

$$a_r = -\omega\zeta \qquad , \quad a_i = \omega\sqrt{1-\zeta^2}$$

and for the first order difference scheme

$$\lambda = x \pm iz$$

where

$$x_c = \frac{\left(1 - \frac{\bar{\omega}}{2}\right)}{\left(1 + \zeta\bar{\omega}\right)} \qquad \text{and} \qquad z_c = \frac{\bar{\omega}\sqrt{1-\zeta^2-\frac{1}{4}\bar{\omega}^2}}{1+\zeta\bar{\omega}}$$

The solution λ can also be given in the form

$$\lambda = r e^{i\theta}$$

where

$$r = \sqrt{x^2 + z^2} \qquad \text{and} \qquad \theta = ARC\ SIN\ \frac{z}{r}$$

A further simplification can be made by replacing r with e^{-K}. Thus,

$$K = -\ln r$$

and λ becomes

$$\lambda = e^{-K} e^{\pm i\theta} \tag{4.8}$$

Examining Eq. 4.7 and 4.8 reveals that

$$a_r\Delta t \quad vs \quad -K$$

and

$$a_i\Delta t \quad vs \quad \theta$$

are proper comparisons. Let the relative error in both K and θ be defined by

$$\mu = \frac{\theta}{a_i\Delta t} \qquad \text{and} \qquad \nu = \frac{-K}{a_r\Delta t}$$

Applying the value of a_i and a_r to μ and ν, leads to

$$\mu = \frac{\theta}{\bar{\omega}\sqrt{1-\zeta^2}} \qquad \text{and} \qquad \nu = \frac{K}{\zeta\bar{\omega}} \tag{4.9}$$

The relative error terms, μ and ν , provide an indication of the accuracy in the numerical solution. This can be seen by comparing the complete solutions. Substituting Eq. 4.3 into Eq. 4.1 yields the exact solution

$$y = A\, e^{\left[-\zeta\omega \pm i\omega\sqrt{1-\zeta^2}\right] J\Delta t} \tag{4.10}$$

since $t = J\Delta t$. Substituting Eq. 4.8 into Eq. 4.4 yields the solutions for the finite difference approximations

$$y = A\, e^{\left[-K \pm i\theta\right] J} \tag{4.11}$$

Applying the relative error terms μ and ν given by Eq. 4.9 to Eq. 4.11, the finite difference solutions take the form

$$y = A\, e^{\left[-\nu\zeta\omega \pm i\mu\omega\sqrt{1-\zeta^2}\right] J\Delta t} \tag{4.12}$$

Thus, a comparison of Eq. 4.12 with 4.10 shows that if μ and ν are both equal to 1.0, Eq. 4.12 is the exact solution.

Examining Eq. 4.12 reveals that μ and ν have a specific role in the error analysis. In Eq. 4.12, $-\nu\zeta\omega$ is a measure of the rate of attenuation in the finite difference solution. Likewise, $\mu\omega\sqrt{1-\zeta^2}$ is a measure of the frequency of the oscillations in the finite difference solution. Therefore, ν will be an indication of the error in attenuation, and μ will be an indication of the error in frequency of the numerical solution.

For the overdamped problem e^{-K} does not change. However, $e^{i\theta}$ becomes an attenuating factor and Eq. 4.12 becomes

$$y = A\, e^{\left[-\nu\zeta\omega \pm \mu\omega\sqrt{\zeta^2-1}\right] J\Delta t}$$

where ν is given in Eq. 4.9, and μ becomes

$$\mu = \frac{i\,\theta}{\bar{\omega}\sqrt{\zeta^2-1}} \tag{4.13}$$

The expressions for λ , κ , θ , and $'\theta$ for Eqs. 3.4, 3.10, and 3.14 are derived in Appendix D. Table I lists the values of λ , κ , and θ , and Table II lists the values for $'\theta$ for all three numerical approximations. The subscripts c, 1, and 2 refer to conventional, MOD 1, and MOD 2 respectively.

In conclusion, a proper comparison of the solutions to the three finite difference approximations with the exact solution has been shown to take the form

$$\nu = \frac{\kappa}{\zeta\,\bar{\omega}} \qquad \text{and} \qquad \mu = \frac{\theta}{\bar{\omega}\sqrt{1-\zeta^2}}$$

for the underdamped problem and

$$\nu = \frac{\kappa}{\zeta\,\bar{\omega}} \qquad \text{and} \qquad \mu = \frac{\theta}{\bar{\omega}\sqrt{\zeta^2-1}}$$

for the overdamped problem. A computer program that makes the necessary comparisons was written in the FORTRAN IV language for the IBM 360 digital computer. The results obtained from this program follow.

TABLE I

LIST OF THE UNDERDAMPED SOLUTIONS

$$\lambda_c = \frac{\left(1 - \frac{\bar{\omega}^2}{2}\right) \pm i\bar{\omega}\sqrt{1 - \zeta^2 - \frac{1}{4}\bar{\omega}^2}}{(1 + \zeta\bar{\omega})}$$

$$\kappa_c = \pm \ln\left(\sqrt{\frac{1 - \zeta\bar{\omega}}{1 + \zeta\bar{\omega}}}\right)$$

$$\theta_c = \text{ARC SIN} \pm \left(\frac{\bar{\omega}^2 - \zeta^2\bar{\omega}^2 - \frac{1}{4}\bar{\omega}^4}{1 - \zeta^2\bar{\omega}^2}\right)^{\frac{1}{2}}$$

$$\lambda_1 = \frac{\left(1 - \frac{\bar{\omega}^2}{3}\right) \pm i\bar{\omega}\sqrt{1 - \zeta^2 - \frac{1}{12}\bar{\omega}^2}}{\left(1 - \zeta\bar{\omega} + \frac{\bar{\omega}^2}{6}\right)}$$

$$\kappa_1 = \pm \ln\left(\frac{\left(1 + \frac{1}{3}\bar{\omega}^2 + \frac{1}{36}\bar{\omega}^4 - \zeta^2\bar{\omega}^2\right)^{1/2}}{\left(1 + \zeta\bar{\omega} + \frac{\bar{\omega}^2}{6}\right)}\right)$$

$$\theta_1 = \text{ARC SIN} \pm \left(\frac{\bar{\omega}^2 - \zeta^2\bar{\omega}^2 - \frac{1}{12}\bar{\omega}^4}{1 - \zeta^2\bar{\omega}^2 + \frac{1}{3}\bar{\omega}^2 + \frac{1}{36}\bar{\omega}^4}\right)^{1/2}$$

$$\lambda_2 = \frac{\left(1 - \frac{5}{12}\bar{\omega}^2\right) \pm i\bar{\omega}\sqrt{1 - \zeta^2 - \frac{1}{6}\bar{\omega}^2}}{\left(1 + \zeta\bar{\omega} + \frac{\bar{\omega}^2}{12}\right)}$$

$$\kappa_2 = \pm \ln\left(\frac{\left(1 + \frac{1}{6}\bar{\omega}^2 + \frac{1}{144}\bar{\omega}^4 - \zeta^2\bar{\omega}^2\right)^{1/2}}{\left(1 + \zeta\omega + \frac{\bar{\omega}^2}{12}\right)}\right)$$

$$\theta_2 = \text{ARC SIN} \pm \left(\frac{\bar{\omega}^2 - \zeta^2\bar{\omega}^2 - \frac{1}{6}\bar{\omega}^4}{1 + \frac{1}{6}\bar{\omega}^2 + \frac{1}{144}\bar{\omega}^4 - \zeta^2\bar{\omega}^2}\right)$$

TABLE II

LIST OF THE OVERDAMPED SOLUTIONS

$$i\theta_c = \text{ARC SINH} \pm \left(\frac{\bar{\omega}^2\zeta^2 - \bar{\omega}^2 + \frac{1}{4}\bar{\omega}^4}{1 - \zeta^2\bar{\omega}^2} \right)^{1/2}$$

$$i\theta_1 = \text{ARC SINH} \pm \left(\frac{\bar{\omega}^2\zeta^2 - \bar{\omega}^2 + \frac{1}{12}\bar{\omega}^4}{1 - \zeta^2\bar{\omega}^2 + \frac{1}{3}\bar{\omega}^2 + \frac{1}{36}\bar{\omega}^4} \right)^{1/2}$$

$$i\theta_2 = \text{ARC SINH} \pm \left(\frac{\bar{\omega}^2\zeta^2 - \bar{\omega}^2 + \frac{1}{6}\bar{\omega}^4}{1 + \frac{1}{6}\bar{\omega}^2 + \frac{1}{144}\bar{\omega}^4 - \zeta^2\bar{\omega}^2} \right)^{1/2}$$

CHAPTER V

RESULTS AND CONCLUSIONS

Results

Figure 2 is a plot of the data in Table III. For this plot, ζ is equal to 0.0, and the system is undamped. The undamped solution will have no attenuation and therefore, the ν term is meaningless.

With ζ = 0.0, the differential equation becomes

$$my'' + ky = 0 \tag{5.1}$$

Recall that the MOD 2 method was derived by maintaining the Hermitian form of the second derivative. As a result, the MOD 2 method produces very little error for all values of $\bar{\omega}$. Since the first derivative is not present in Eq. 5.1, the MOD 1 method can be expected to have more error than MOD 2. However, it is found that the error in MOD 1 is approximately the same as the error associated with the conventional method.

The significant gain in accuracy by the employment of MOD 2 means a larger time interval could be used to approximate the system without loss of accuracy. For example, if the natural frequency of the system equaled 180 radians per second, a time interval of 0.005 seconds would describe the system with 0.2 per cent error in using MOD 2. To achieve this accuracy with the conventional method, a time interval of 0.0015 seconds must be used. This means that the time interval can be increased by a factor of three by using the MOD 2 approximation for this problem.

As the damping is increased, the coefficient of the first derivative increases, and the first derivative begins to have a significant influence upon the response. The MOD 1 method should become more accurate as ζ increases since it is derived by maintaining the Hermitian form of the first derivative. Figures 3 and 4 are plots of the data presented in Table IV with ζ equal to 0.9. Since ζ is less than 1.0, the underdamped comparisons are used. As anticipated,

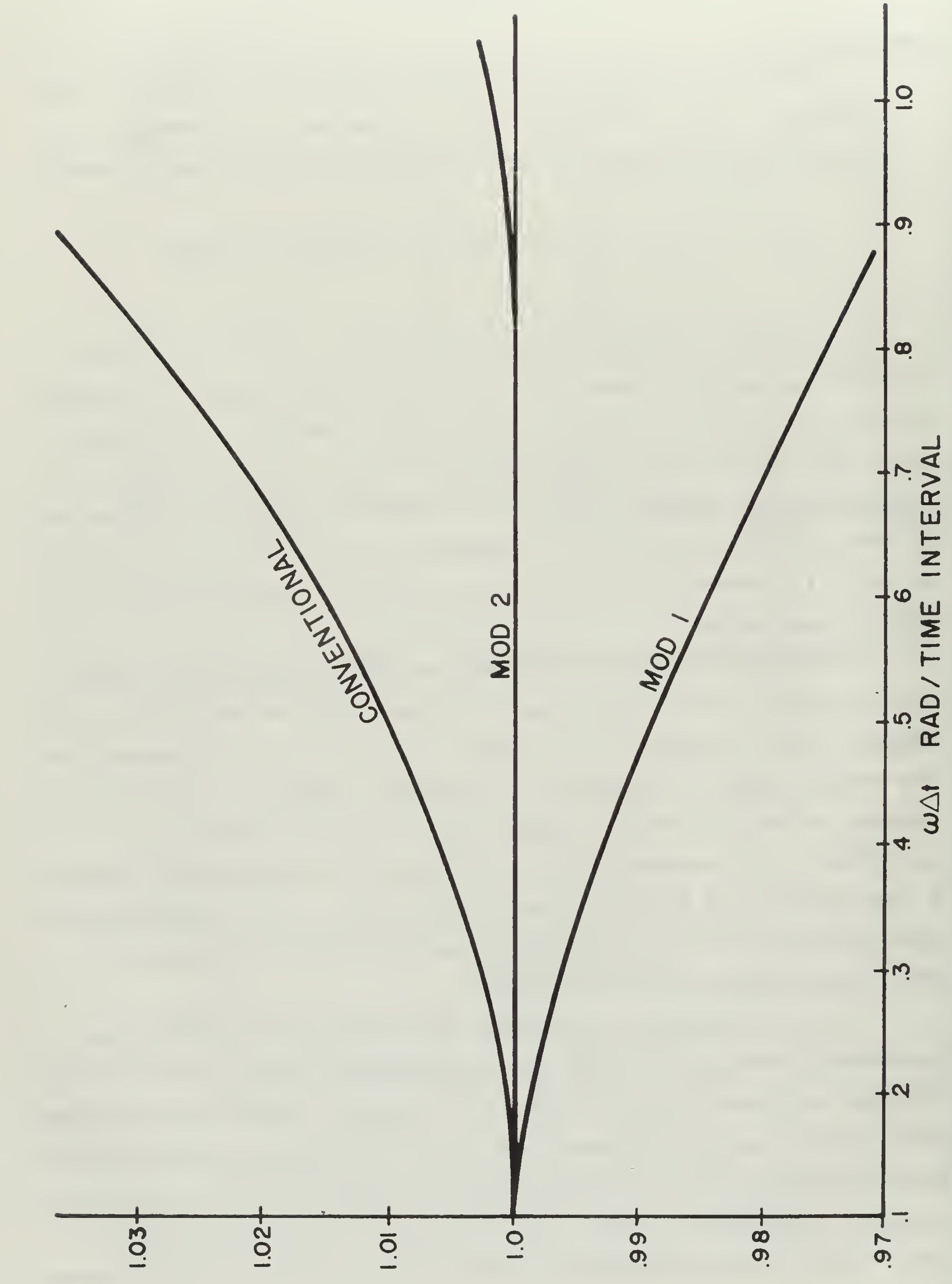

FIGURE 2 μ vs SAMPLING RESOLUTION FOR DAMPING = 0.0

TABLE III

RELATIVE ERROR FOR $\zeta = 0.0$

ω	μ_c	μ_1	μ_2
0.1	1.00042	0.99958	1.00000
0.2	1.00167	0.99834	1.00000
0.3	1.00379	0.99629	1.00002
0.4	1.00679	0.99345	1.00005
0.5	1.01072	0.98987	1.00013
0.6	1.01564	0.98559	1.00027
0.7	1.02163	0.98065	1.00051
0.8	1.02879	0.97513	1.00088
0.9	1.03726	0.96907	1.00142
1.0	1.04720	0.96255	1.00219

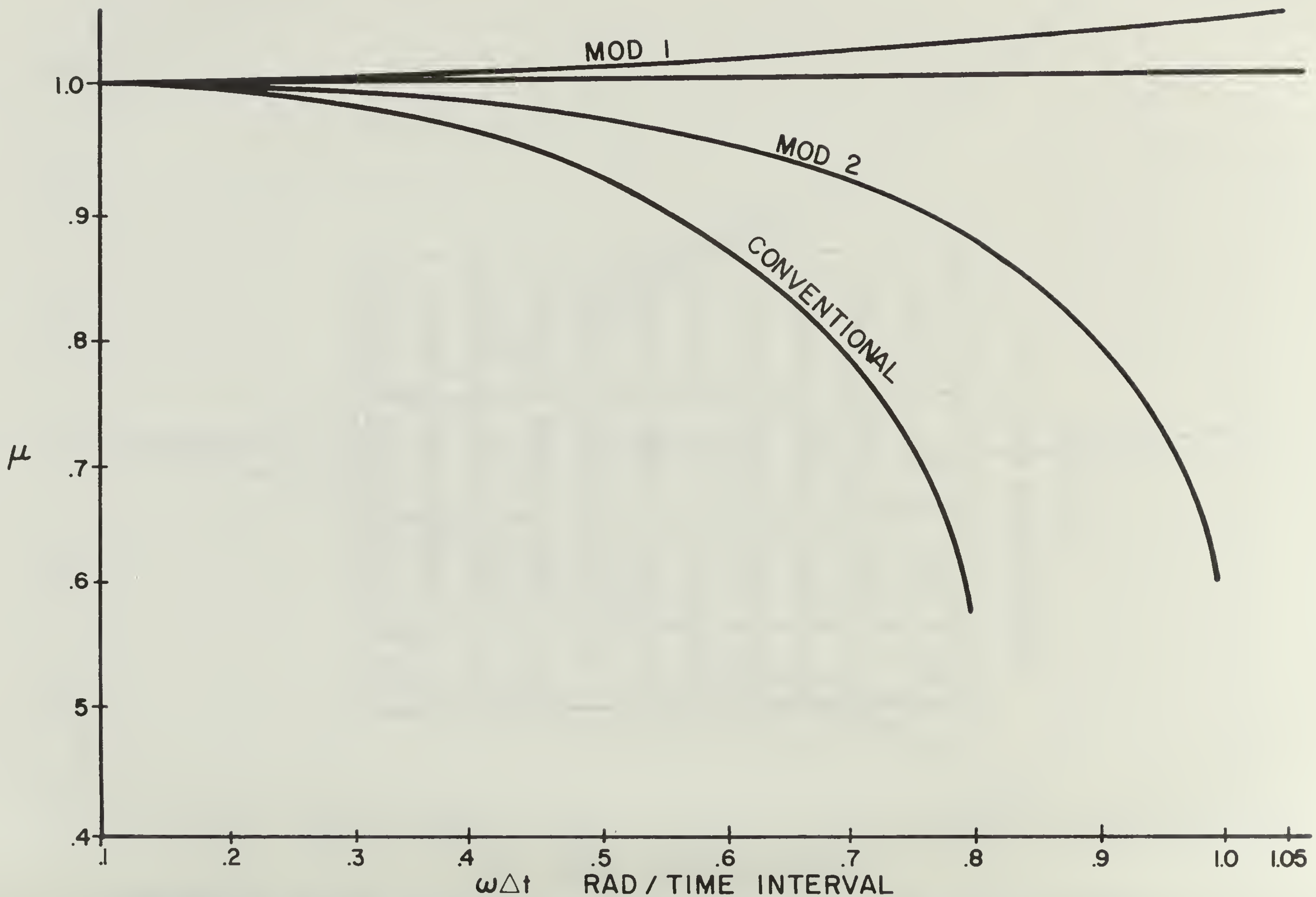

FIGURE 3 μ vs SAMPLING RESOLUTION FOR DAMPING = 0.9

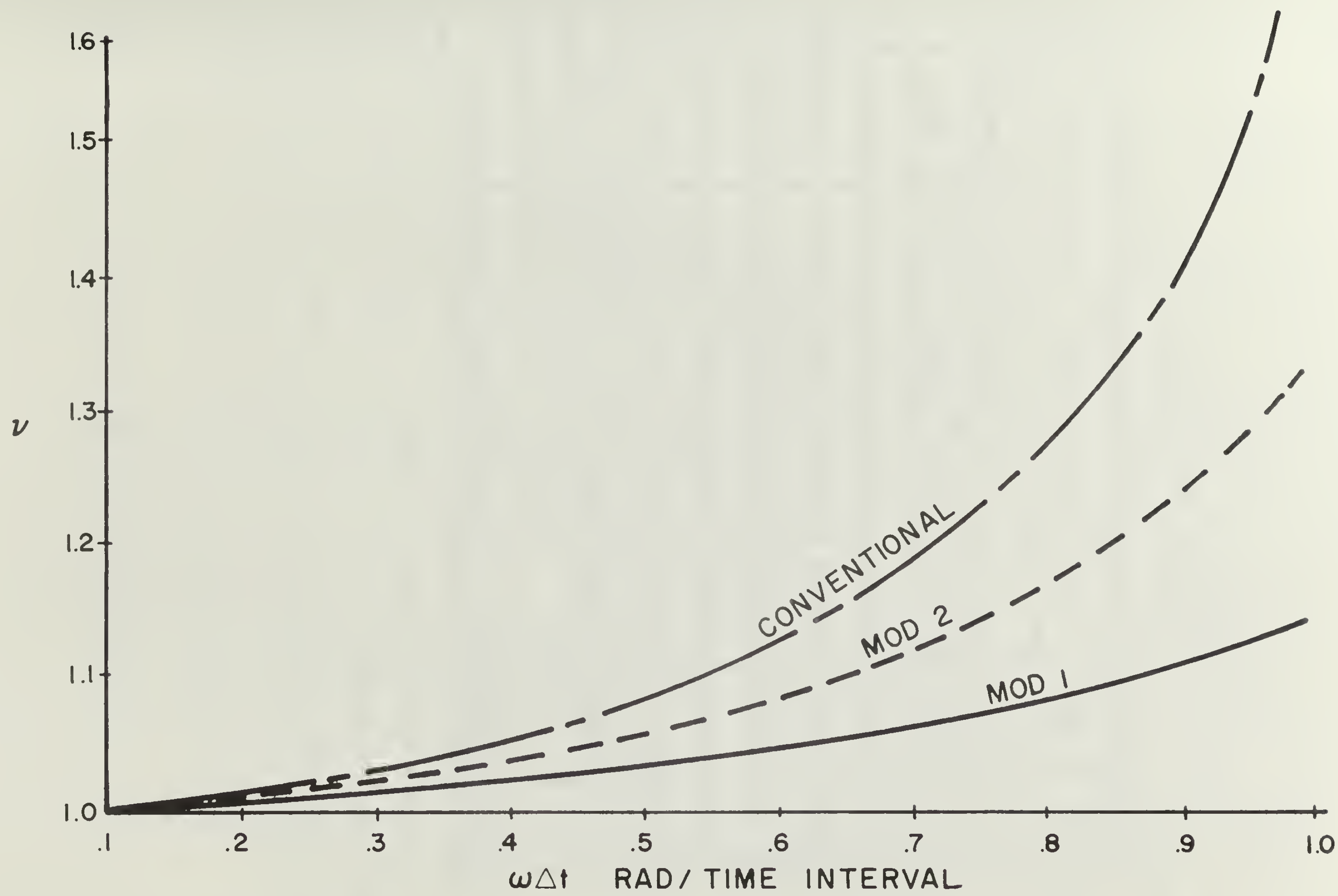

FIGURE 4 ν vs SAMPLING RESOLUTION FOR DAMPING = 0.9

TABLE IV

RELATIVE ERROR FOR $\zeta = 0.9$

$\bar{\omega}$	ν_c	μ_c	ν_1	μ_1	ν_2	μ_2
0.1	1.0033	0.99776	1.0022	1.00051	1.0022	0.99914
0.2	1.0116	0.99072	1.0044	1.00203	1.0078	0.99643
0.3	1.0248	0.97780	1.0096	1.00456	1.0174	0.99151
0.4	1.0466	0.95681	1.0169	1.00809	1.0319	0.98364
0.5	1.0771	0.92353	1.0273	1.01260	1.0515	0.97153
0.6	1.1190	0.86943	1.0409	1.01804	1.0781	0.95291
0.7	1.1770	0.77470	1.0571	1.02432	1.1133	0.92357
0.8	1.2605	0.57647	1.0785	1.03123	1.1608	0.87490
0.9	1.3913	0.0	1.1044	1.03841	1.2259	0.78621
1.0	1.6360	0.0	1.1375	1.04507	1.3230	0.58758

the MOD 1 approximation is seen to be the most accurate. For example, in Figure 3 at a resolution of 0.8 the MOD 1 approximation has 3 per cent error in θ. For this same resolution, the conventional approximation has 42 per cent relative in θ.

Figures 5 and 6 are plots of the relative error versus ζ for a constant resolution of 0.5 as listed in Table V. Figure 5 shows that one of the two Hermitian approximations always has a better value of μ than the conventional approximation. For low values of ζ the MOD 2 approximation is best, and for high values of ζ the MOD 1 approximation is best. This means that throughout the range of ζ, greater accuracy in the frequency can be obtained by using one of the Hermitian approximations.

The error in attenuation, ν, is shown in Figure 6. Note that for the lightly damped system, where the attenuation is small, the conventional approximation has the least error. However, this does not imply that the total solution given by Eq. 4.12 is most accurate when the conventional approximation is employed since Figure 5 shows that the conventional approximation has the largest error in μ for the lightly damped system. For values of ζ between 0.45 and 0.65, the MOD 2 approximation is best, and for values of ζ greater than 0.65, the MOD 1 approximation is best. Thus, Figure 6 shows that the Hermitian approximations have less error in attenuation for values of ζ greater than 0.45.

Analysis was not made for the critically damped oscillator. However, as ζ approaches 1.0 from either side, Figure 5 shows that μ becomes very large. In order to understand what happens to the numerical solution for $.9 < \zeta < 1.1$, it is necessary to make other comparisons. For example, the exact solution, Eq. 4.10, and the solution for the finite difference approximations, Eq. 4.12, can both be given in the following form

$$y = A \, e^{\left[-\zeta \pm i\sqrt{1-\zeta^2}\right]\omega \Delta t J}$$

and

$$y = A \, e^{\left[-\zeta\nu \pm i\mu\sqrt{1-\zeta^2}\right]\omega \Delta t J}$$

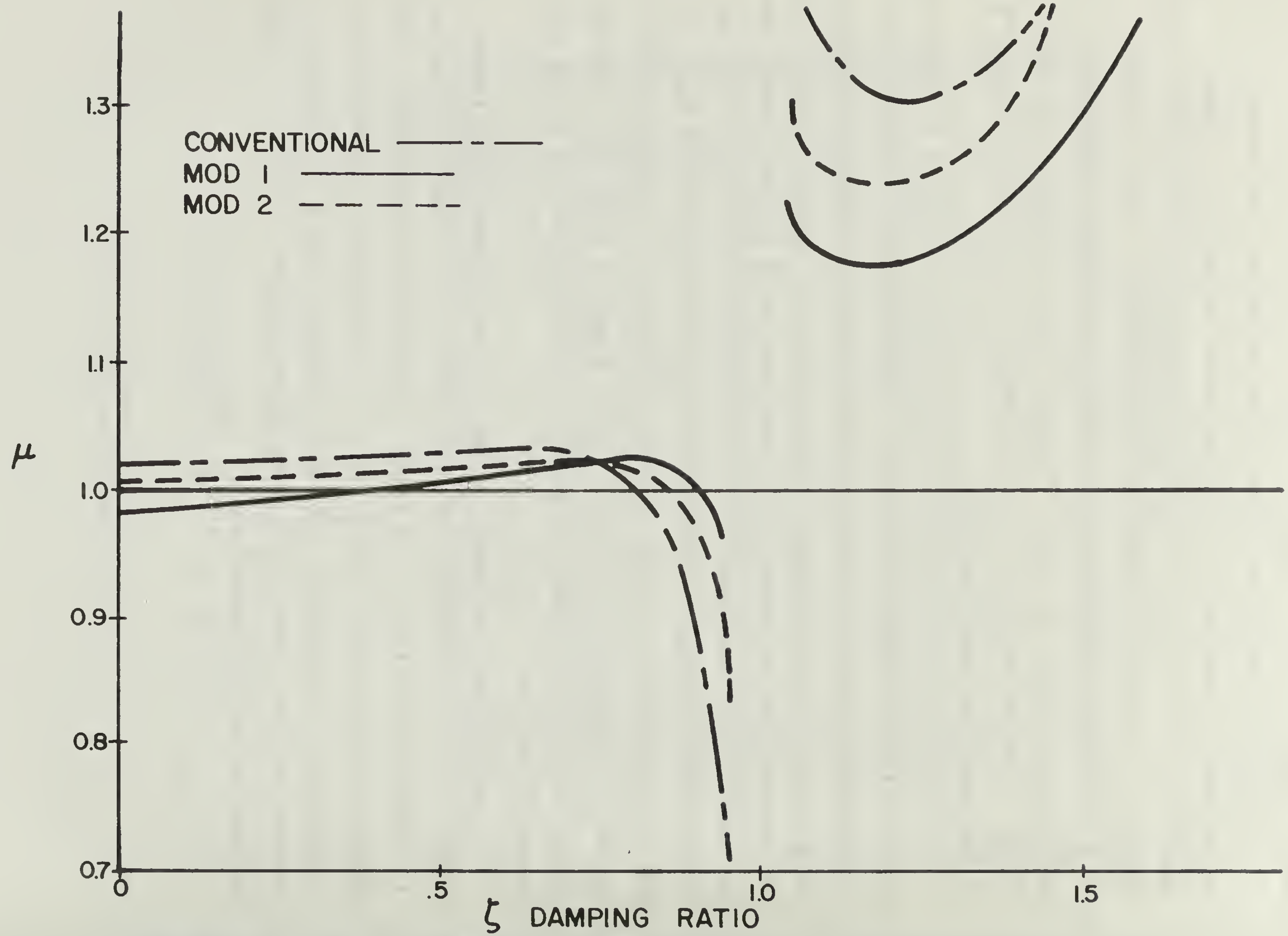

FIGURE 5 μ vs DAMPING FOR SAMPLING RESOLUTION = 0.5

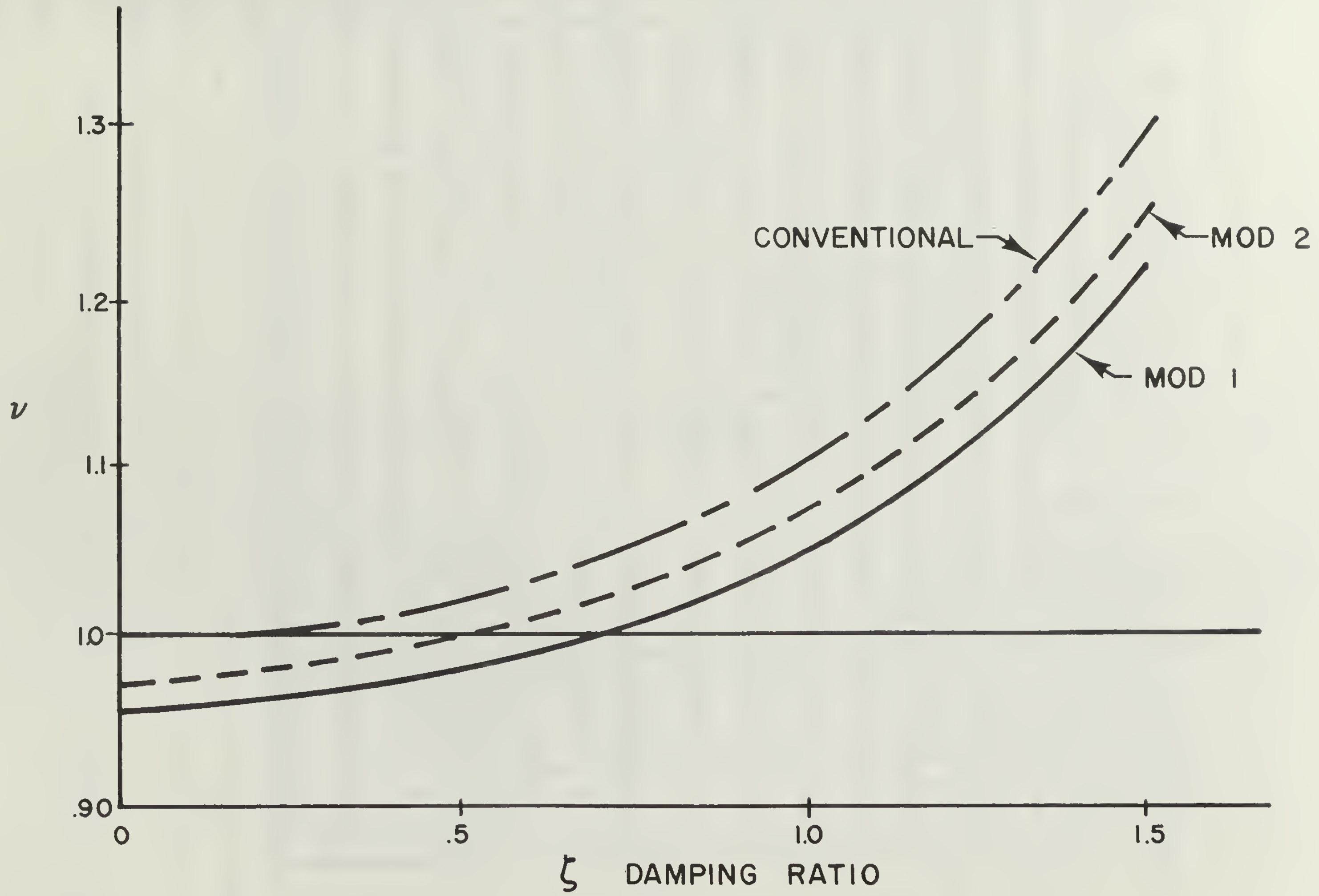

FIGURE 6 ν vs DAMPING FOR SAMPLING RESOLUTION = 0.5

TABLE V

RELATIVE ERROR FOR $\bar{\omega} = 0.5$

ζ	ν_c	μ_c	ν_1	μ_1	ν_2	μ_2
0	-	1.01072	-	0.98987	-	1.00013
.1	1.000	1.01125	.9600	0.99055	.9780	1.00074
.2	1.004	1.01282	.9630	0.99259	.9830	1.00255
.3	1.008	1.01531	.9666	0.99598	.9873	1.00551
.4	1.0140	1.01849	.9725	1.00070	.9920	1.00948
.5	1.022	1.02184	.9792	1.00665	1.0004	1.01418
.6	1.0317	1.02419	.9880	1.01356	1.0096	1.01890
.7	1.0440	1.02254	.9988	1.02067	1.0211	1.02181
.8	1.0630	1.00712	1.0120	1.02527	1.0350	1.01692
.9	1.0771	0.92353	1.0273	1.01260	1.0515	0.97153
1.1	1.1244	1.34268	1.0680	1.17106	1.0950	1.25556
1.2	1.1552	1.29560	1.0940	1.17212	1.1257	1.23259
1.3	1.1926	1.30888	1.1252	1.19704	1.1578	1.25391
1.4	1.2392	1.34629	1.1634	1.23407	1.1998	1.30399
1.5	1.2970	1.40661	1.2101	1.28394	1.2517	1.42483

for $\zeta < 1.0$. Thus a plot of $\sqrt{1-\zeta^2}$ and $\mu\sqrt{1-\zeta^2}$ versus ζ shows the error in the oscillatory part of the solution for the underdamped system. For the overdamped system, $\zeta > 1.0$, Eqs. 4.10 and 4.12 become

$$y = A\, e^{\left[-\zeta \pm \sqrt{\zeta^2-1}\right]\omega\Delta t\, J}$$

and

$$y = A\, e^{\left[-\nu\zeta \pm \mu\sqrt{\zeta^2-1}\right]\omega\Delta t\, J}$$

Thus, for this range of ζ a plot of $\sqrt{\zeta^2-1}$ and $\mu\sqrt{\zeta^2-1}$ versus ζ will be the equivalent comparison for the overdamped system.

Examination of the equations in Table I reveals that the numerical solution will be critically damped for the values of ζ slightly less than 1.0. Therefore, to show the critical damping point for each finite difference approximation it is necessary to use the comparisons for the overdamped system for values of $.95 < \zeta < 1.0$. Equation 4.13 defines μ for the overdamped system as

$$\mu = \frac{i\theta}{\bar{\omega}\sqrt{\zeta^2-1}}$$

therefore,

$$\mu\sqrt{\zeta^2-1} = \frac{i\theta}{\bar{\omega}} \qquad (5.2)$$

Using Eq. 5.2, it is possible to evaluate the overdamped comparisons for values of $\zeta < 1.0$.

Figure 7 is a plot of $\mu\sqrt{1-\zeta^2}$ or $\mu\sqrt{\zeta^2-1}$ for values of ζ between 0.9 and 1.1 with $\omega = 0.5$. The MOD 1 approximation has its critical damping point closest to 1.0, and the conventional approximation has its critical damping point furthest from 1.0. As $\bar{\omega}$ is decreased and the system is sampled more often, the critical damping points of all the numerical solutions will approach 1.0. However, the Hermitian approximations will always be critically damped closer to 1.0 than the conventional approximation. The fact that each numerical solution has its own critical damping point that is less than 1.0 explains the large values of μ in this region.

For the overdamped system a problem arises when $\zeta\bar{\omega}$ is greater than 1.0. Examination of the equations in Tables I and II reveals an unstable zone for these finite difference equations at this point. Although $\zeta\bar{\omega} = 1.0$ is essentially the limit for numerical stability,

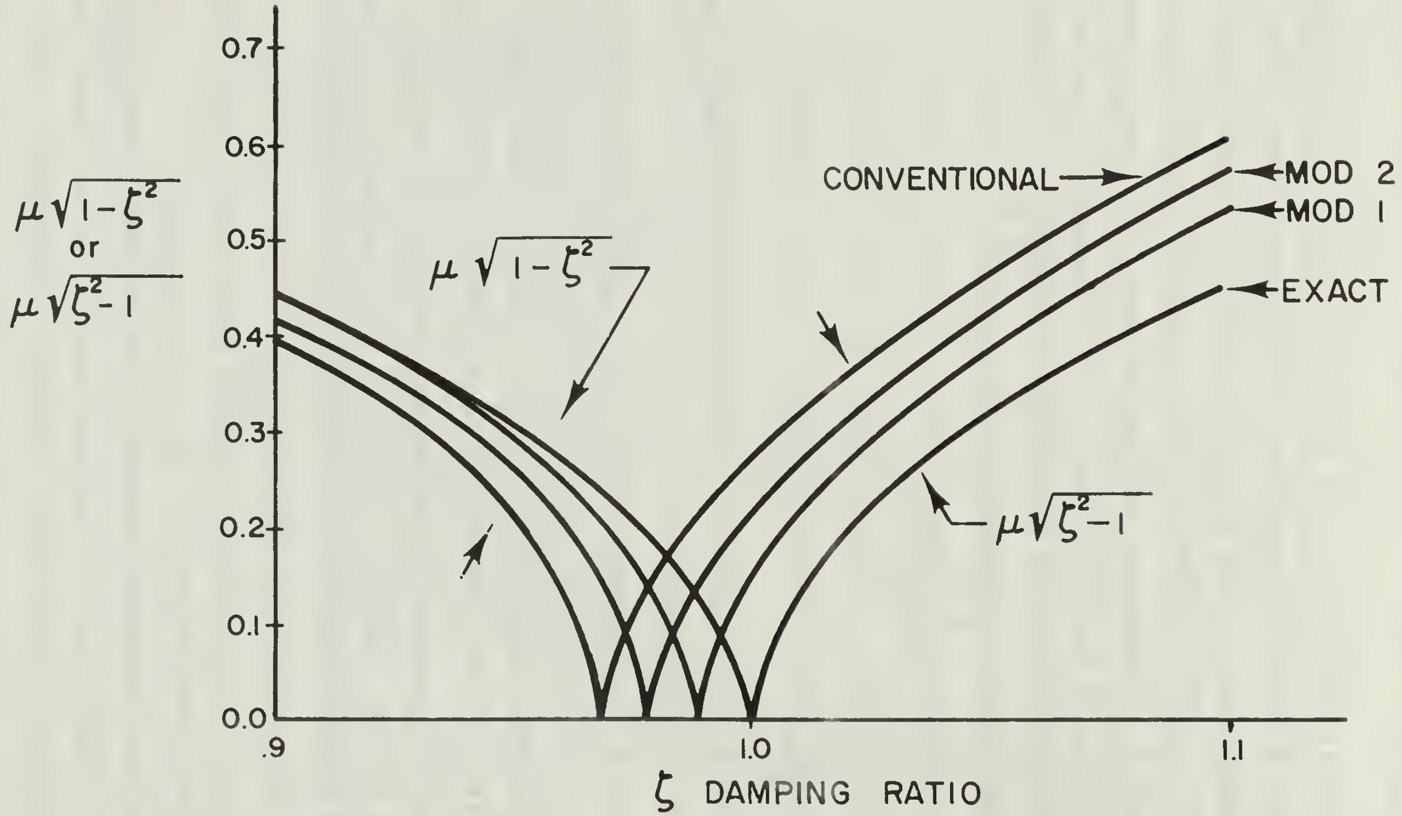

FIGURE 7 INVESTIGATION OF THE REGION NEAR CRITICAL DAMPING

the accuracy of the difference approximations becomes very poor before the unstable zone is reached. As the stability limit is approached, both μ and ν approach infinity. Figure 8 shows the stability limits on all three finite difference approximations. This plot was made using the equations in Tables I and II.

Conclusions

From the results presented here, the conclusion can be made that the Hermitian equations can be used to obtain more accurate numerical solutions for the same size increment than can be obtained by first order central finite difference equations when applied to differential equations containing derivatives of more than one degree. Further, this increase in accuracy is not accompanied either by an increase in complexity of the resulting matrix when applied to boundary value problems or by a significant increase in computation time when applied to initial value problems. Consequently, use of the Hermitian equations would be most advantageous when computer core storage space is limited and accurate results are required.

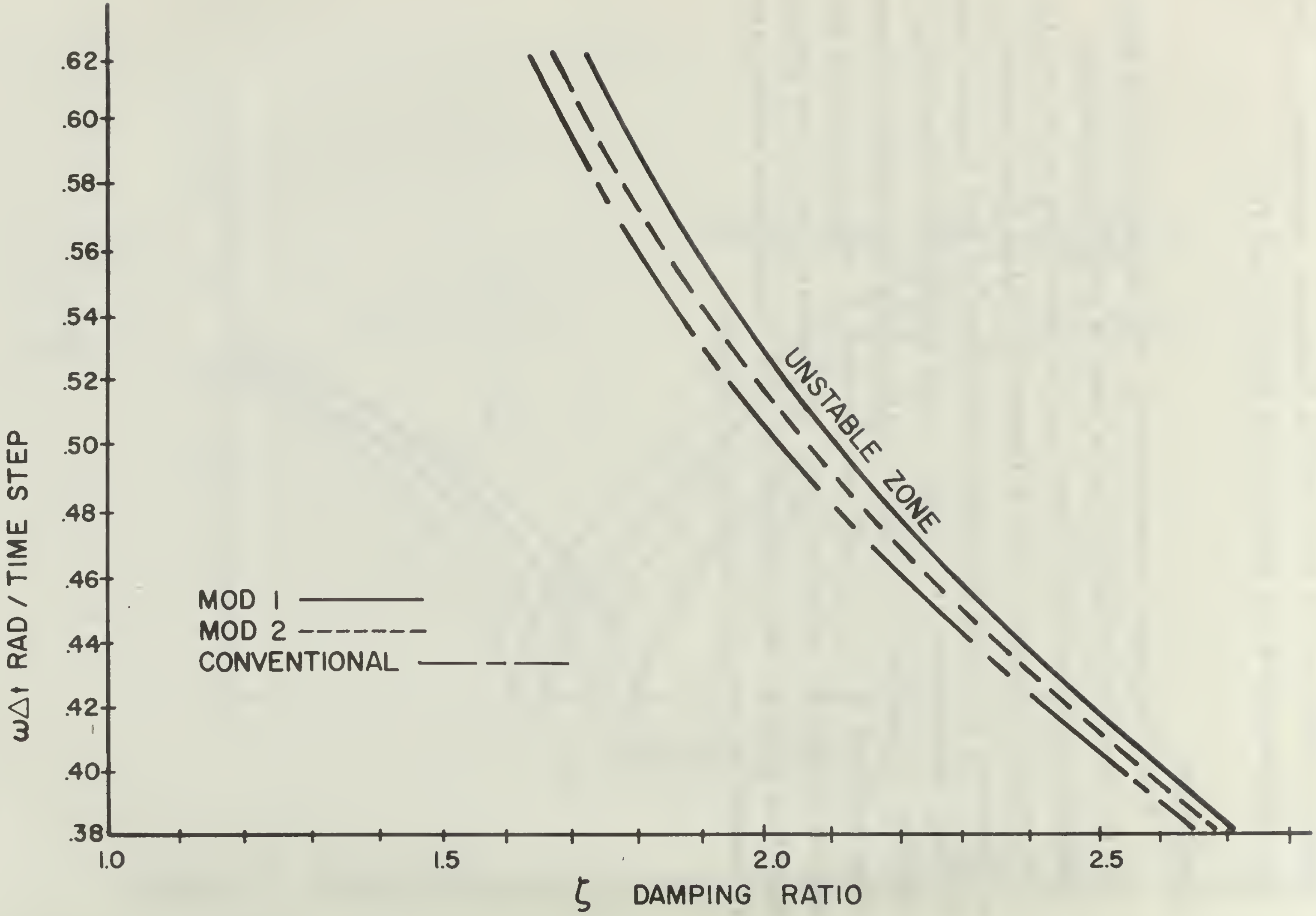

FIGURE 8 DAMPING vs SAMPLING RESOLUTION NEAR STABILITY LIMIT

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APPENDIX A

THE DERIVATION OF THE CONVENTIONAL EQUATIONS FOR THE FIRST AND SECOND DERIVATIVE

The first order central finite difference equations can be derived from the Taylor series. The stations are lableled station j+1, station j, and station j-1. The interval between stations j+1 and j is called Δ .

The Taylor series for y_{j+1} is

$$y_{j+1} = y_j + \Delta y_j' + \frac{\Delta^2}{2!} y_j'' + \frac{\Delta^3}{3!} y_j''' + \frac{\Delta^4}{4!} y_j^{IV} + \frac{\Delta^5}{5!} y_j^{V} + \cdots \tag{A.1}$$

For y_{j-1}, the Taylor series becomes

$$y_{j-1} = y_j - \Delta y_j' + \frac{\Delta^2}{2!} y_j'' - \frac{\Delta^3}{3!} y_j''' + \frac{\Delta^4}{4!} y_j^{IV} - \frac{\Delta^5}{5!} y_j^{V} + \cdots \tag{A.2}$$

Subtracting Eq. A.1 from Eq. A.2 gives

$$y_{j+1} - y_{j-1} = 2\Delta y_j' + \frac{\Delta^3}{3} y_j''' + \frac{\Delta^5}{60} y_j^{V} + \cdots \tag{A.3}$$

Solving for y'_j leads to

$$y_j' = \frac{1}{2\Delta}\left(y_{j+1} - y_{j-1}\right) - \frac{\Delta^2}{6} y_j''' - \frac{\Delta^4}{120} y_j^{V} + \cdots \tag{A.4}$$

This is the first order central finite difference equation for the first derivative.

Adding Eqs. A.1 and A.2 gives

$$y_j'' = \frac{1}{\Delta^2}\left(y_{j+1} - 2y_j + y_{j-1}\right) - \frac{\Delta^2}{12} y_j^{IV} - \frac{\Delta^4}{360} y_j^{VI} + \cdots \tag{A.5}$$

Equation A.5 is the first order central finite difference equation for the second derivative.

The truncation error in each equation is usually approximated by the first term in the Taylor series that is not used in the difference approximation. For example, the error in Eq. A.4 is

$$\frac{\Delta^2}{6} y_J'''$$

and the error in Eq. A.5 is

$$\frac{\Delta^2}{12} y_J^{IV}$$

APPENDIX B

THE DERIVATION OF THE HERMITIAN EQUATIONS FOR THE FIRST AND SECOND DERIVATIVES

Taking the derivative of Eq. A.1 in Appendix A gives

$$y'_{J+1} = y'_J + \Delta\, y''_J + \frac{\Delta^2}{2} y'''_J + \frac{\Delta^3}{6} y^{IV}_J + \frac{\Delta^4}{24} y^{V}_J + \frac{\Delta^5}{120} y^{VI}_J + \cdots \tag{B.1}$$

Likewise Eq. A.2 becomes

$$y'_{J-1} = y'_J - \Delta\, y''_J + \frac{\Delta^2}{2} y'''_J - \frac{\Delta^3}{6} y^{IV}_J + \frac{\Delta^4}{24} y^{V}_J \quad \frac{\Delta^5}{120} y^{VI}_J + \cdots \tag{B.2}$$

Adding Eqs. B.1 and B.2 gives

$$y'_{J+1} + y'_{J-1} = 2 y'_J + \Delta^2 y'''_J + \frac{\Delta^4}{12} y^{V}_J + \cdots \tag{B.3}$$

Dividing Eq. B.3 by 6 and solving for $\frac{\Delta^2}{6}$ y''' leads to

$$\frac{\Delta^2}{6} y'''_J = \frac{1}{6}\left(y'_{J+1} - 2 y'_J + y'_{J-1}\right) - \frac{\Delta^4}{72} y^{V}_J + \cdots \tag{B.4}$$

From Eq. A.4 of Appendix A,

$$y'_J = \frac{1}{2\Delta}\left(y_{J+1} - y_{J-1}\right) - \frac{\Delta^2}{6} y'''_J - \frac{\Delta^4}{120} y^{V}_J + \cdots \tag{A.4}$$

Thus, substituting Eq. B.4 into Eq. A.4 gives

$$y'_J = \frac{1}{2\Delta}\left(y_{J+1} - y_{J-1}\right) - \frac{1}{6}\left(y'_{J+1} - 2 y'_J + y'_{J-1}\right) + \frac{\Delta^4}{180} y^{V}_J + \cdots \tag{B.5}$$

Solving for $y_{j+1} - y_{j-1}$ gives

$$y_{j+1} - y_{j-1} = \frac{\Delta}{3}\left(y'_{j+1} + 4y'_j + y'_{j-1}\right) - \frac{\Delta^5}{90} y_j^{V} + \cdots \tag{B.6}$$

Equation B.6 is the Hermitian equation for the first derivative.

The second derivative of equation A.1 is

$$y''_{j+1} = y''_j + \Delta\, y'''_j + \frac{\Delta^2}{2} y_j^{IV} + \frac{\Delta^3}{6} y_j^{V} + \frac{\Delta^4}{24} y_j^{VI} + \frac{\Delta^5}{120} y_j^{VII} + \cdots \tag{B.7}$$

Likewise Eq. A.2 becomes

$$y''_{j-1} = y''_j - \Delta\, y'''_j + \frac{\Delta^2}{2} y_j^{IV} - \frac{\Delta^3}{6} y_j^{V} + \frac{\Delta^4}{24} y_j^{VI} - \frac{\Delta^5}{120} y_j^{VII} + \cdots \tag{B.8}$$

Adding Eqs. B.7 and B.8 gives

$$y''_{j+1} + y''_{j-1} = 2y''_j + \Delta^2 y_j^{IV} + \frac{\Delta^4}{12} y_j^{VI} + \cdots \tag{B.9}$$

Dividing Eq. B.9 by 12 and solving for $\frac{\Delta^2}{12} y^{IV}$ yields

$$\frac{\Delta^2}{12} y_j^{IV} = \frac{1}{12}\left(y''_{j+1} - 2y''_j + y''_{j-1}\right) - \frac{\Delta^4}{144} y_j^{VI} + \cdots \tag{B.10}$$

From Eq. A.5 of Appendix A,

$$y''_j = \frac{1}{\Delta^2}\left(y_{j+1} - 2y_j + y_{j-1}\right) - \frac{\Delta^2}{12} y_j^{IV} - \frac{\Delta^4}{360} y_j^{VI} + \cdots \tag{A.5}$$

Substituting Eq. B.10 into Eq. A.5 gives

$$y''_j = \frac{1}{\Delta^2}\left(y_{j+1} - 2y_j + y_{j-1}\right) - \frac{1}{12}\left(y''_{j+1} - 2y''_j + y''_{j-1}\right) + \frac{\Delta^4}{240} y_j^{VI} + \cdots \tag{B.11}$$

Then, solving for $(y_{j+1} - 2y_j + y_{j-1})$ gives

$$\left(y_{j+1} - 2y_j + y_{j-1}\right) = \frac{\Delta^2}{12}\left(y''_{j+1} + 10y''_j + y''_{j-1}\right) - \frac{\Delta^6}{240} y_j^{VI} + \cdots \tag{B.12}$$

Equation B.12 is the Hermitian equation for the second derivative.

A comparison of the conventional equations in Appendix A with the Hermitian equations derived here reveals that the Hermitian equations have less truncation error. This is to be expected since the error term in each first order equation was used to derive the respective Hermitian equation.

APPENDIX C

DISCUSSION OF THE TRUNCATION ERROR IN THE MODIFIED HERMITIAN EQUATION

In the MOD 2 method of analysis, the Hermitian equation for the first derivative is modified. Equation B.6 is the Hermitian equation for the first derivative

$$y_{J+1} - y_{J-1} = \frac{\Delta}{3}\left(y'_{J+1} + 4y'_J + y'_{J-1}\right) - \frac{\Delta^5}{90} y_J^{V} \tag{B.6}$$

This equation can be written

$$\frac{c}{4\Delta}\left(y_{J+1} - y_{J-1}\right) = \frac{c}{12}\left(y'_{J+1} + 4y'_J + y'_{J-1}\right) - \frac{\Delta^5}{360} y_J^{V} \tag{C.1}$$

The conventional equation for y'_j can be written

$$\frac{c}{4\Delta}\left(y_{J+1} - y_{J-1}\right) = \frac{c}{2} y'_J + \frac{\Delta^2}{12} y_J''' \tag{C.2}$$

Addition of equations C.1 and C.2 gives

$$\frac{c}{2\Delta}\left(y_{J+1} - y_{J-1}\right) = \frac{c}{12}\left(y'_{J+1} + 10y'_J + y'_{J-1}\right) + \frac{\Delta^2}{12} y_J''' - \frac{\Delta^5}{360} y_J^{V} \tag{C.3}$$

Equation C.3 is the modified Hermitian equation for the first derivative that was used in the MOD 2 approximation. The truncation error in this equation is

$$\frac{\Delta^2}{12} y_J'''$$

This truncation error is one-half the truncation error in the first order equation for the first derivative. Therefore; in the MOD 2 approximation to the differential equation, there is very little gain in accuracy associated with the first derivative.

Likewise, for the MOD 1 approximation, the Hermitian equation for the second derivative must be modified. The modified equation is

$$\frac{m}{\Delta^2}\left(y_{J+1} - 2y_J + y_{J-1}\right) = \frac{m}{6}\left(y''_{J+1} + 4y''_J + y''_{J-1}\right) - \frac{\Delta^2}{12}y_J^{IV} - \frac{\Delta^6}{90}y_J^{VI} \tag{C.4}$$

The truncation error,

$$\frac{\Delta^2}{12}\, y_J^{IV}$$

is equal to the truncation error of the first order difference equation for the second derivative. Therefore, in the MOD 1 approximation, there is no gain in accuracy associated with the second derivative.

APPENDIX D

SOLUTIONS FOR THE FINITE DIFFERENCE APPROXIMATIONS

The conventional approximation will be used as an example for this appendix. The conventional finite difference expression for the differential equation is

$$\frac{m}{(\Delta t)^2}\left(y_{J+1} - 2y_J + y_{J-1}\right) + \frac{c}{2\Delta t}\left(y_{J+1} - y_{J-1}\right) + k y_J = 0 \qquad \text{(D.1)}$$

When the assumed solution

$$y_J = A\lambda^J$$

is substituted into Eq. D.1, the result is

$$\left[\frac{m}{(\Delta t)^2}\left(\lambda_c - 2 - \lambda_c^{-1}\right) + \frac{c}{2\Delta t}\left(\lambda_c - \lambda_c^{-1}\right) + k\right] A\lambda_c^J = 0$$

For an arbitrary solution,

$$\frac{m}{(\Delta t)^2}\left(\lambda_c^2 - 2\lambda_c + 1\right) + \frac{c}{2\Delta t}\left(\lambda_c^2 - 1\right) + k\lambda_c = 0$$

or

$$\left[\frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}\right]\lambda_c^2 + \left[k - \frac{2m}{(\Delta t)^2}\right]\lambda_c + \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right] = 0$$

By using the quadratic equation,

$$\lambda_c = \frac{-\left[k - \frac{2m}{(\Delta t)^2}\right] \pm \sqrt{\left[k - \frac{2m}{(\Delta t)^2}\right]^2 - 4\left[\frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}\right]\left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right]}}{2\left[\frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}\right]}$$

or

$$\lambda_c = \frac{\left|2m - k(\Delta t)^2\right| \pm \sqrt{\left[2m - k(\Delta t)^2\right]^2 - 4\left[m^2 - \frac{1}{4}c^2(\Delta t)^2\right]}}{2\left[m - \frac{c}{2}\Delta t\right]}$$

When the substitutions

$$\omega = \sqrt{\frac{k}{m}}\ , \quad \zeta = \frac{c}{2m\omega} \quad \text{and} \quad \bar{\omega} = \omega\Delta t$$

are made, λ_c becomes

$$\lambda_c = \frac{\left[1 - \frac{1}{2}\bar{\omega}^2\right] \pm \bar{\omega}\sqrt{\zeta^2 - 1 + \frac{1}{4}\bar{\omega}^2}}{\left[1 + \zeta\bar{\omega}\right]}$$

To make λ_c complex, take $\sqrt{-1}$ outside the radical. Then

$$\lambda_c = \frac{\left[1 - \frac{1}{2}\bar{\omega}^2\right] \pm i\omega\sqrt{1 - \zeta^2 - \frac{1}{4}\bar{\omega}^2}}{\left[1 + \zeta\bar{\omega}\right]} \qquad \text{(D.2)}$$

The solution λ_c is now of the form

$$\lambda_c = x + iz$$

To get λ_c into the form

$$\lambda_c = r\,e^{i\theta}$$

the following relationships are used.

$$r_c = \sqrt{x^2 + z^2} \quad \text{and} \quad \theta_c = \text{ARC SIN}\ \frac{z}{r_c}$$

Therefore,

$$r_c = \frac{\pm\left(\left[1 - \frac{1}{2}\bar{\omega}^2\right]^2 + \bar{\omega}^2\left[1 - \zeta^2 - \frac{1}{4}\bar{\omega}^2\right]\right)^{1/2}}{1 + \zeta\bar{\omega}}$$

or $$r_c = \pm\sqrt{\frac{1 - \zeta\bar{\omega}}{1 + \zeta\bar{\omega}}}$$

if, $K = -\ln r$, then

$$K_c = -\ln \pm\sqrt{\frac{1 - \zeta\bar{\omega}}{1 + \zeta\bar{\omega}}} \tag{D.3}$$

Similarly,

$$\theta_c = \text{ARC SIN} \pm \sqrt{\frac{\bar{\omega}^2 - \zeta^2\bar{\omega}^2 - \frac{1}{4}\bar{\omega}^4}{1 - \zeta^2\bar{\omega}^2}} \tag{D.4}$$

For the overdamped system, i θ_c must be found. From Eq. D.4,

$$\text{SIN}\ \theta_c = \pm\sqrt{\frac{\bar{\omega}^2 - \zeta^2\bar{\omega}^2 - \frac{1}{4}\bar{\omega}^4}{1 - \zeta^2\bar{\omega}^2}}$$

Then,

$$i\ \text{SIN}\ \theta_c = \pm\sqrt{\frac{-\bar{\omega}^2 + \zeta^2\bar{\omega}^2 + \frac{1}{4}\bar{\omega}^4}{1 - \zeta^2\bar{\omega}^2}} = \text{SINH}\ i\theta_c$$

since

$$i\ \text{SIN}\ \theta = \text{SINH}\ i\theta$$

Therefore,

$$i\theta = \text{ARC SINH} \pm \sqrt{\frac{\zeta^2\bar{\omega}^2 - \bar{\omega}^2 + \frac{1}{4}\bar{\omega}^2}{1 - \zeta^2\bar{\omega}^2}} \tag{D.5}$$

The equations for λ, K, θ, and iθ for MOD 1 and MOD 2 are derived in a similar manner.

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13. ABSTRACT

Approximation of the differential equation of the single degree of freedom, damped oscillator by the first order central finite difference equations and two Hermitian finite difference equations is investigated. An error analysis is made between the solution to the differential equation and the solutions to the finite difference approximations for various values of damping. The results of the error analysis indicate that the Hermitian approximations have less error than the first order difference approximation for the same size of increment. Furthermore, the employment of the Hermitian method does not materially increase the execution time on the digital computer over that required by the first order difference equations. Thus the Hermitian equations are superior to the first order finite difference equations for the damped oscillator problem.

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